06.03.2023

TUTORIAL - 4

1. The joint PDF of (X, Y) is given by $f(x,y) = k(2x+3y)$, $x = 0,1,2$, $y = 1,2,3$

Sol.

X \ Y	1	2	3	$P\{X=x_i\}$
0	3k	6k	9k	18k
1	5k	8k	11k	24k
2	7k	10k	13k	30k
$P\{Y=y_j\}$	15k	24k	33k	72k.

$$\sum_i \sum_j P_{ij} = 1$$

$$72k = 1$$

$$k = 1/72$$

X \ Y	1	2	3	$P\{X=x_i\}$
0	3/72	6/72	9/72	18/72
1	5/72	8/72	11/72	24/72
2	7/72	10/72	13/72	30/72
$P\{Y=y_j\}$	15/72	24/72	33/72	1

i) Marginal density of X

X	$P\{X=x_i\}$
0	~~3/15~~ 18/72
1	~~5/15~~ 24/72
2	~~7/15~~ 30/72
Total	1

Marginal density of Y

Y	$P\{Y=y_j\}$
1	15/72
2	24/72
3	33/72
Total	1

ii) Conditional probability distribution

X	$P\{X=x_i/Y=1\}$	
0	$\frac{P_{01}}{P_{*1}} = \frac{3/72}{15/72}$	3/15
1	$\frac{P_{11}}{P_{*1}} = \frac{5/72}{15/72}$	5/15
2	$\frac{P_{21}}{P_{*1}} = \frac{7/72}{15/72}$	7/15

X	$P\{X=x_i/Y=2\}$
0	$\frac{6/72}{24/72} = 6/24$
1	$\frac{8/72}{24/72} = 8/24$
2	$\frac{10/72}{24/72} = 10/24$

X	$P\{X=x_i \mid Y=3\}$
0	$\frac{9/72}{33/72} = 9/33$
1	$\frac{11/72}{33/72} = 11/33$
2	$\frac{13/72}{33/72} = 13/33$

y	$P\{Y=y_j \mid x=0\}$	
1	$\frac{3/72}{18/72} = 3/18$	$\frac{P_{01}}{P_{0*}}$
2	$\frac{6/72}{18/72} = 6/18$	$\frac{P_{02}}{P_{0*}}$
3	$\frac{9/72}{18/72} = 9/18$	$\frac{P_{03}}{P_{0*}}$

y	$P\{Y=y_j \mid x=1\}$
1	$\frac{15/72}{24/72} = 15/24$
2	$\frac{8/72}{24/72} = 8/24$
3	$\frac{11/72}{24/72} = 11/24$

y	$P\{Y=y_j \mid x=2\}$
1	$\frac{7/72}{30/72} = 7/30$
2	$\frac{10/72}{30/72} = 10/30$
3	$\frac{13/72}{30/72} = 13/30$

1,2 1,2

iii) $P(X \geq 1, Y \leq 2)$.

$$P(X\geq 1, Y\leq 2) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1) + P(X=2, Y=2)$$

$$= \frac{5}{72} + \frac{8}{72} + \frac{7}{72} + \frac{10}{72}$$

$$= \frac{30}{72}$$

iv) $P(X+Y \leq 4)$

X+Y	P(X+X)
1	3/72
2	11/72
3	24/72
4	21/72
5	13/72
T	1

$X+Y \leq 4$	$P(X+Y\leq 4)$	
1	3/72	(0,1) = 1
2	11/72	(0,2), (1,1), (2,1)
3	24/72	(0,3) (1,2) (2,1)
4	21/72	
T	59/72	

2. If the joint pdf of (X,Y) is $f(x,y) = kxye^{-(x^2+y^2)}$, $x>0, y>0$

i) Find k.

W.K.T $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$\int_0^{\infty}\int_0^{\infty} kxye^{-(x^2+y^2)}\,dx\,dy = 1$$

$$k\int_0^{\infty} xe^{-x^2}\,dx * \int_0^{\infty} ye^{-y^2}\,dy = 1$$

Let $x^2 = u$ Let $y^2 = v$

$2x\,dx = du$ $2y\,dy = dv$

$x\,dx = \frac{du}{2}$ $y\,dy = \frac{dv}{2}$

x	0	∞
u	0	∞

$u \to 0$ to ∞

y	0	∞
v	0	∞

$v \to 0$ to ∞

$$k\int_0^\infty e^{-u}\frac{du}{2}\int_0^\infty e^{-v}\frac{dv}{2} = 1$$

$$\frac{k}{4}(-e^{-u})_0^\infty(-e^{-v})_0^\infty = 1$$

$$\frac{k}{4}(-e^{-\infty}+e^{0})(-e^{-\infty}+e^{0}) = 1$$

$$\frac{k}{4}(0+1)(0+1) = 1$$

$$k = 4$$

$$\therefore f(x,y) = 4xye^{-(x^2+y^2)}, \quad x>0, y>0.$$

ii) Prove that X & Y are independent

Let $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^\infty 4xye^{-(x^2+y^2)}\,dy$

$$= 4x\int_0^\infty ye^{-x^2}e^{-y^2}\,dy = 4xe^{-x^2}\int_0^\infty ye^{-y^2}\,dy$$

$$= 4xe^{-x^2}\int_0^\infty e^{-v}\frac{dv}{2} = 2xe^{-x^2}(-e^{-v})_0^\infty$$

$$= 2xe^{-x^2}(0+1)$$

$$\therefore f(x) = 2xe^{-x^2}, \quad x>0$$

Let $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^\infty 4xye^{-(x^2+y^2)}\,dx$

$$= \int_0^\infty 4xye^{-x^2}e^{-y^2}\,dx = 4ye^{-y^2}\int_0^\infty xe^{-x^2}\,dx$$

$$= 4ye^{-y^2}\int_0^\infty e^{-u}\frac{du}{2} = 2ye^{-y^2}(-e^{-u})_0^\infty$$

$$f(y) = 2ye^{-y^2}, \quad y>0.$$

Now $f(x)\times f(y) = 2xe^{-x^2}\times 2ye^{-y^2}; \quad x>0, y>0$

$$= 4xye^{-(x^2+y^2)}, \quad x>0, y>0$$

$$= f(x,y)$$

Hence, X & Y are independent

3. If $f(x,y) = 8xy, 0<x<1, 0<y<x$. Find the marginal density of X.

Sol:

Marginal density of X.

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^x 8xy\,dy$$

$$= \overset{4}{\cancel{8}}x \left[\frac{y^2}{\cancel{2}}\right]_0^x$$

$$= 4x\,(x^2 - 0)$$

$$f(x) = 4x^3, \quad 0<x<1$$

8/3/23